

Infinite Volume of Noncommutative Black Hole Wrapped by Finite Surface

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The volume of a black hole under noncommutative spacetime background is found to be infinite, in contradiction with the surface area of a black hole, or its Bekenstein-Hawking (BH) entropy, which is well-known to be finite. Our result rules out the possibility of interpreting the entropy of a black hole by counting the number of modes wrapped inside its surface if the final evaporation stage can be properly treated. It implies the statistical interpretation for the BH entropy can be independent of the volume, provided spacetime is noncommutative. The effect of radiation back reaction is found to be small and doesn't influence the above conclusion.

PACS numbers: 04.70.Dy, 04.62.+v, 02.40.Gh

I. INTRODUCTION

The interior of a black hole is not causally connected to its exterior. As a result an external observer is prohibited from gaining any information about the interior of a black hole, including the value of its volume. The volume of a black hole was first investigated by Parikh [1]. Subsequent studies invoked a slicing invariant definition with different time-like Killing vectors [2–6]. Recently, Christodoulou and Rovelli (CR) suggested a different approach [7], which was followed by investigations in several spacetime backgrounds [8–10]. The CR volume V_{CR} is defined as the largest one bounded by event horizon. Such a definition paints a different picture for the final stage of a collapsed black hole, allowing it to possess a very large interior instead of shrinking to a point.

The volume of a black hole is often discussed in connection with the information loss paradox [11]. The life of a black hole is often considered to consist of two distinct segments: formation followed by evaporation, assuming the latter starts only after the former is completed. Such a hypothesis prompts a natural but seldomly asked question: where is the information about the collapsed matter at the instant separating the two segments? i.e., when formation is completed but evaporation is yet to begin.

For Schwarzschild black hole, plausible answers consider information stored in its interior, or distributed on its horizon as was initially postulated in the “quantum hairs” discussion [12]. The existence of “quantum hairs” implies that information could reside on horizon before evaporation. Recently, Hawking *et al.* show how to implant soft hairs on the horizon with soft degrees of freedom proportional to the area of horizon in Planck units, based on physical processes of the elegant mechanism using the soft graviton theorem [13]. The possibility that information can hide in the interior, however, is not ruled out. To establish a firm answer, Parikh suggested the illuminating idea [1] along the conversation between Jacobson, Marolf, and Rovelli [14] to construct families of spacetime whose horizon areas or surfaces are bounded but whose volumes wrapped inside can be arbitrarily large. The entropies for such black holes must be in-

dependent of their volumes. The amount of information associated with the Bekenstein-Hawking (BH) entropies would have to be distributed over their horizons.

For stationary black holes in both three and four dimensions, Parikh did not provide any preconceived constructions [1]. Christodoulou and Rovelli show $V_{\text{CR}} \sim 3\sqrt{3}\pi M^2 v$ for a Schwarzschild black hole [7], with M the (initial) mass and v the advanced time. This interesting result satisfies the requirement of Parikh [1], i.e., the volume becomes infinite for $v \rightarrow \infty$. However, due to Hawking radiation [15], a Schwarzschild black hole evaporates and disappears at $v \sim M^3$. Such an estimate for v limits the volume for the (initial) black hole to a large but finite value $V_{\text{CR}} \sim M^5$.

Whether this volume is sufficient to house enough modes for explaining the BH entropy statistically? A recent calculation [16] implicates a negative answer, although it finds the entropy calculated by counting modes housed in this volume is proportional to the surface area. However, its treatment for the final evaporation stage might be inadequate [16]. An improved treatment would likely lead to a revised mass loss rate as evaporation approaches the Planck scale, which will alter the estimated lifetime of a black hole. This Letter presents our effort towards this by invoking spacetime noncommutativity, which is capable of treating the final evaporation stage without difficult singularities [17, 18].

Adoption of spacetime noncommutativity leads to different black holes [19, 20] and their associated thermodynamics (see [21] for a review and the references therein). In noncommutative spacetime, the singularity in the interior of a black hole disappears, and a remnant always arises irrespective [22, 23] after black hole evaporation, which helps to remove the so-called Hawking paradox of a diverging temperature as the black hole radius shrinks to zero. We show such an approach overcomes the uncertainty of the earlier result [16] by providing an improved description for the final evaporation stage, with which we establish a firm answer to whether the interior of a noncommutative black hole is large enough to explain the BH entropy. We find a concrete example for a black hole with an infinite volume but a finite horizon area. Throughout

this paper, we use units with $G = c = \hbar = k_B = 1$.

II. NONCOMMUTATIVE BLACK HOLE

We begin by briefly reviewing the noncommutative Schwarzschild black hole and its thermodynamics. The spacetime coordinate x^μ becomes noncommuting [17]

$$[x^\mu, x^\nu] = i\theta\epsilon^{\mu\nu}, \quad (1)$$

with the noncommutative parameter θ of dimension length squared. It is a constant required by the Lorentz invariance and unitarity [24]. $\epsilon^{\mu\nu}$ is a real anti-symmetric tensor. The commutation relation Eq. (1) gives $\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}\theta$, the analogous Heisenberg uncertainty relationship, which dictates the spacetime to be “point-less”, free from gravitational singularity.

The direct application of noncommutative coordinates to black holes is inconvenient, so in this paper, we adopt the idea assumed in Ref. [17] where the spatial noncommutative effect is attributed to the modified energy-momentum tensor as a source while the Einstein tensor is not changed.

In flat spacetime noncommutativity eliminates point-like structures in favor of smeared distributions [19, 20]. When applied to spacetime in gravity, one can simply make corresponding substitutions. Instead of the Dirac δ -function $\rho_\theta(r) = M\delta(r)$ usually employed for a point mass M at origin in commutative spacetime, the smearing leads to a Gaussian distribution

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{\frac{3}{2}}} e^{-\frac{r^2}{4\theta}}, \quad (2)$$

of a width $\sim \sqrt{\theta}$, with which the energy-momentum tensor was identified for a self-gravitating droplet of anisotropic fluid [17]. Solving the Einstein equation gives

$$ds^2 = - \left[1 - \frac{4M}{r\sqrt{\pi}} \gamma \left(\frac{3}{2}, \frac{r^2}{4\theta} \right) \right] dt^2 + \left[1 - \frac{4M}{r\sqrt{\pi}} \gamma \left(\frac{3}{2}, \frac{r^2}{4\theta} \right) \right]^{-1} dr^2 + r^2 d\Omega^2, \quad (3)$$

with the lower incomplete gamma function $\gamma(\nu, x) = \int_0^x t^{\nu-1} e^{-t} dt$, which approaches $\sqrt{\pi}/2$ as $r \rightarrow \infty$. For $\theta \rightarrow 0$, $\gamma(\nu, x)$ reduces to the usual $\Gamma(\nu)$ -function and the noncommutative metric Eq. (3) becomes the commutative Schwarzschild metric.

The condition of $g_{tt}(r_h) = 0$ gives the event horizon

$$r_h = \frac{4M}{\sqrt{\pi}} \gamma \left(\frac{3}{2}, \frac{r_h^2}{4\theta} \right) \equiv \frac{4M}{\sqrt{\pi}} \gamma_h, \quad (4)$$

which takes the minimum $r_h^{(\min)} = r_0 \simeq 3.0\sqrt{\theta}$ at $M_0 \simeq 1.9\sqrt{\theta}$ determined by $dM/dr_h = 0$ where the two horizons decay to one. In particular, no horizon exists below M_0 , which is not concerned in our paper.

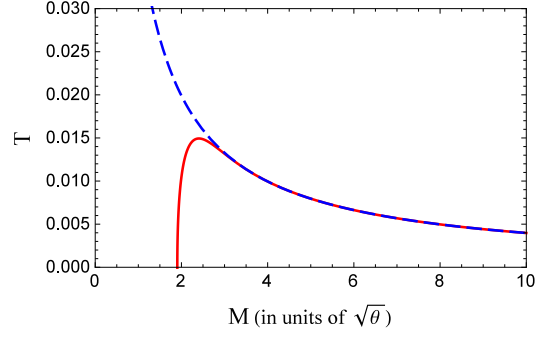


FIG. 1. (Color online) The temperatures T for an evaporating Schwarzschild black hole of (initial) mass M . The red-solid (blue-dashed) line refers to noncommutative (commutative) spacetime.

The temperature is obtained for the static noncommutative metric Eq. (3),

$$T_h = \frac{1}{4\pi} \frac{dg_{tt}}{dr} \Big|_{r=r_h} = \frac{1}{4\pi r_h} \left(1 - \frac{r_h^3}{4\theta^{\frac{3}{2}} \gamma_h} e^{-\frac{r_h^2}{4\theta}} \right), \quad (5)$$

which reaches its maximum at $M \simeq 2.4\sqrt{\theta}$ and decreases to zero at $M = M_0$ as shown by the red-solid line in Fig. 1. When $\theta \rightarrow 0$, T_h reduces to $T_H = 1/(8\pi M)$, as for commutative spacetime denoted by the blue-dashed line. From the first law of black hole thermodynamics $TdS = dM$, we find the entropy

$$S_h = \int \frac{dM}{T} \simeq 4\pi M^2 \left(1 - \frac{4M}{\sqrt{\pi}\theta} e^{-\frac{M^2}{\theta}} \right), \quad (6)$$

up to the order of $e^{-\frac{M^2}{\theta}}/\sqrt{\theta}$.

III. VOLUME AND ENTROPY

It was shown [7, 25] that the definition of CR volume can be applied to Schwarzschild black holes, but also applied to any other spherically symmetric spacetime. Therefore, as described in last section, the interior of noncommutative black holes (3) can define the CR volume. For this, one has to rewrite the metric (3) in terms of ingoing Eddington-Finkelstein coordinates,

$$ds^2 = -f(r) dv^2 + 2dvdr + r^2 d\varphi^2 + r^2 \sin^2 \varphi d\phi^2, \quad (7)$$

with $f(r) = 1 - \frac{4M}{r\sqrt{\pi}} \gamma \left(\frac{3}{2}, \frac{r^2}{4\theta} \right)$, and the advanced time $v = t + r^*$ for $r^* \equiv \int^r \frac{dr'}{f(r')}$. As shown before [7, 8], the CR volume is mostly related to the region which is not causally connected with matter that has fallen far into the black hole, and so the contribution for the volume is given by the integral

$$V_{\text{NCR}} = \int^v \max[F(r)] dv d\varphi d\phi, \quad (8)$$

where the integral will be dominated by its upper limit v while the lower integration limit is irrelevant, as pointed out in Ref. [8]. Then we calculate the maximal value of the function by setting $\frac{dF}{dr} = 0$ for the integrand $F(r) = r^2 \sqrt{\frac{4M}{r\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) - 1}$, and the maximum is found to occur at

$$r \simeq r_n \left(1 - \frac{r_n}{\sqrt{\pi\theta}} e^{-\frac{r_n^2}{4\theta}}\right), \quad (9)$$

with $r_n = 3M/2$. Carrying out the integration, we find

$$V_{\text{NCR}} \simeq 3\sqrt{3}\pi M^2 v \left(1 - \frac{2r_n}{\sqrt{\pi\theta}} e^{-\frac{r_n^2}{4\theta}}\right), \quad (10)$$

which is explicitly modified by θ , apart from implicit θ -dependence, e.g., in v .

Next we discuss the entropy associated with V_{NCR} . We change the metric Eq. (7) into the form

$$ds^2 = -dT^2 - [f(r)\dot{v}^2 - 2\dot{v}\dot{r}] d\lambda^2 + r^2 d\Omega^2, \quad (11)$$

with the transformation $dv = \frac{-1}{\sqrt{-f}} dT + d\lambda$ and $dr = \sqrt{-f} dT$. Since the volume refers to the late time v at $r \simeq r_n \left(1 - \frac{r_n}{\sqrt{\pi\theta}} e^{-\frac{r_n^2}{4\theta}}\right)$, we take the constant- T hypersurface to count the number of quantum field modes that can be housed in V_{NCR} . With suitable modifications consistent with the uncertainty relationship [26–28], the earlier method [16] remains applicable to more general cases of quantum gravity effects, including spacetime noncommutativity being discussed here. The commutation between conjugate position x_i and momenta p_j is unchanged, or $[x_i, p_j] = i\delta_{ij}$ [19, 20], i.e., the uncertainty relation $\Delta x_i \Delta p_i \sim 2\pi$ retains. Now the phase space is labeled by $\{\lambda, \varphi, \phi, p_\lambda, p_\varphi, p_\phi\}$, and the volume element takes the form $d\lambda d\varphi d\phi dp_\lambda dp_\varphi dp_\phi / (2\pi)^3$.

Compared with the commutative spacetime discussed earlier [16], the only change concerns the factor $f(r)$ when computing noncommutative entropy. We find the same form [16] for,

$$S_{\text{NCR}} = \frac{\pi^2 V_{\text{NCR}}}{45\beta_h^3}, \quad (12)$$

with both V_{NCR} and β_h modified by noncommutativity. To study S_{NCR} in more detail, we need to specify the time v . As shown clearly by the red solid line in Fig. 1, evaporation of a noncommutative black hole involves two stages; before the maximal temperature is reached, it is similar to what happens in commutative Schwarzschild black hole; after that, the temperature decreases to zero and a cold remnant of mass $M_0 \simeq 1.9\sqrt{\theta}$ is left behind.

First, we investigate the influence of back reaction. As well-known, the dynamic evolution for a spherically symmetric black hole due to Hawking radiation can be described by the Vaidya metric [29]. With this metric, the CR volume was already calculated and the change

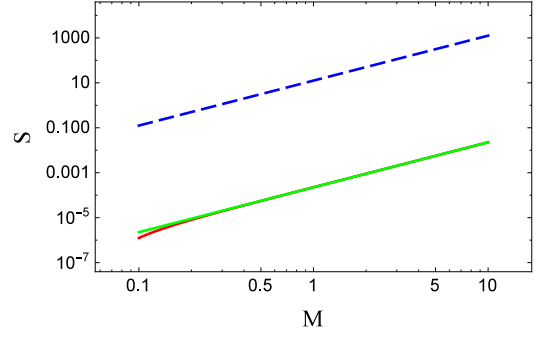


FIG. 2. (Color online) Entropies S for a commutative Schwarzschild black hole (of initial mass M). The red (green) solid line denotes the entropy associated with the volume when back reaction is included (excluded). The blue dashed line denotes the noncommutative BH entropy S_h of Eq. (6).

was found to be insignificant [10, 25], where the estimation for advanced time v was made by using the Stefan-Boltzmann law (see Eq. (13) below) since the black hole was considered with the mass greater than the Planck mass [30]. However, the associated entropy is yet to be investigated for this case.

Transforming the mass into $M'(v) = \Theta(v)(M^3 - 3Bv)^{\frac{1}{3}}$ with $\Theta(v)$ the Heaviside step function and the parameter $B \sim 10^{-3}$ related to the back reaction [30], the CR volume can be reexpressed as [10, 25],

$$V'_{\text{CR}} \simeq 3\sqrt{3}\pi M^2 v \left(1 - \frac{9B}{2M^2}\right),$$

with which we can compute the entropy associated with CR volume including the effect of back reaction using Eq. (12). Figure 2 illustrates clearly that the entropy associated with the volume is insufficient for a statistical interpretation of the Bekenstein-Hawking entropy, even when the back reaction is included. This interesting result refutes against the possibility of balancing black hole information loss by a huge but finite volume [7, 25, 31]. It is also seen from Fig. 2 quantitatively that the influence of back reaction is small for our purpose. In what follows we will not include modifications from the back reaction in our calculations.

According to Ref. [32], in the first evaporation stage, the mass loss rate for a black hole is given by the Stefan-Boltzmann law

$$\frac{dM}{dv} = -\frac{1}{\gamma M^2}, \quad \gamma > 0. \quad (13)$$

The specific value for the constant γ does not influence the present study. Integrating from an initial black hole mass M , one finds $v \sim \gamma(M^3 - M_f^3) \sim \gamma M^3$, with M_f ($\ll M$) being the critical mass where Eq. (13) becomes invalid. Assuming the first evaporation stage dominates v , omitting the second evaporation stage, and with the

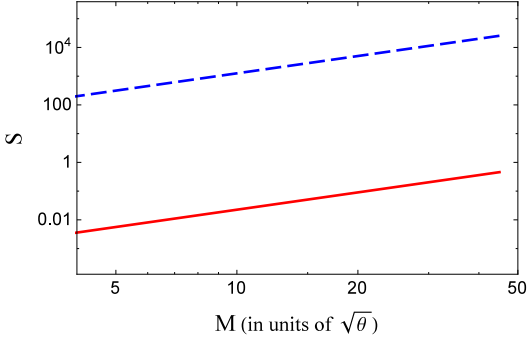


FIG. 3. (Color online) The entropy S associated with the volume of a noncommutative black hole in red solid line with estimated v limited to the first evaporation stage. The blue dashed line denotes the noncommutative BH entropy S_h .

definitions for M and $\beta_h = T_h^{-1}$, we find

$$S_{\text{NCR}} \sim \frac{\sqrt{3}\gamma M^2}{7680} \left(1 - \frac{3Me^{-\frac{9M^2}{16\theta}}}{\sqrt{\pi\theta}}\right) \left(1 - \frac{2Me^{-\frac{M^2}{\theta}}}{\sqrt{\pi\theta}}\right)^3.$$

The surface area for a noncommutative black hole can be expressed approximately as $A_h = 16\pi M^2 \left(1 - \frac{2M}{\sqrt{\pi\theta}}e^{-\frac{M^2}{\theta}}\right)^2$. Thus the entropy S_{NCR} associated with the noncommutative volume is proportional to the horizon area of a noncommutative black hole. We can rewrite it as $S_{\text{NCR}} \sim \frac{\sqrt{3}\gamma\varepsilon(\theta)}{122880\pi} A_h$, where $\varepsilon(\theta) = \left(1 - \frac{3M}{\sqrt{\pi\theta}}e^{-\frac{9M^2}{16\theta}}\right) \left(1 - \frac{2M}{\sqrt{\pi\theta}}e^{-\frac{M^2}{\theta}}\right)$ approaches unity if v is approximated by the first evaporation stage. Figure 3 compares S_{NCR} with S_h . The entropy associated with the noncommutative volume remains clearly insufficient for a statistical interpretation of the BH entropy if v only accounts for the first evaporation stage.

A refined description with noncommutative space-time can incorporate the second evaporation stage [21], which offers a different scenario from the final explosion with a diverging temperature as for a commutative Schwarzschild black hole shown in Fig. 1. Modification due to noncommutative spacetime kicks in at the final evaporation stage, especially when M_f approaches M_0 . In this regime, the temperature can be approximated as $T_h \simeq \alpha(M_f - M_0)$, with $\alpha = \frac{dT_h}{dM}|_{r_h=r_0}$. The same analysis as in Ref. [32] gives for large v ,

$$v \sim \frac{1}{(M_f - M_0)^3}. \quad (14)$$

The final evaporation stage thus needs an infinite time, although the net change to the black hole radius will only be several $\sqrt{\theta}$. While counter-intuitive at first sight, this result is consistent with the third law of thermodynamics: zero temperature state cannot be reached with a countable number of steps or within a finite time. The statistical entropy in Eq. (12) formally remains the same as in commutative spacetime, except for the noncommutative modifications in the expression of the volume. We

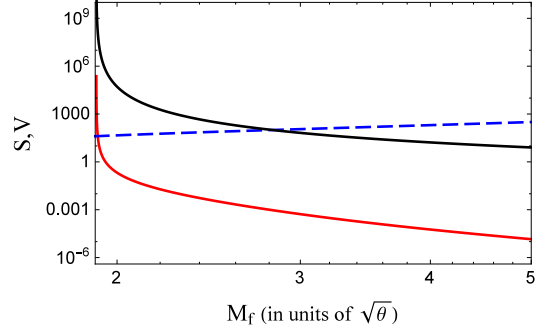


FIG. 4. (Color online) The comparison between the noncommutative volume V (black solid line) and the associated entropy S (red solid line). The blue dashed line denotes the noncommutative BH entropy S_h of Eq. (6).

thus immediately arrive at

$$V_{\text{NCR}} \simeq 3\sqrt{3}\pi \frac{M_f^2 \left(1 - \frac{2r_n}{\sqrt{\pi\theta}}e^{-\frac{r_n^2}{4\theta}}\right)}{(M_f - M_0)^3}, \quad (15)$$

for $v \rightarrow \infty$. It constitutes an example for a black hole with an infinite volume wrapped by a finite horizon since V_{NCR} is divergent when $M_f \rightarrow M_0$, as shown in Fig. 4.

From Eq. (12), we obtain

$$S_{\text{NCR}} \sim \eta \alpha^3 A_h, \quad (16)$$

with $\eta = \frac{\sqrt{3}\pi^2 \left(1 - \frac{2r_n}{\sqrt{\pi\theta}}e^{-\frac{r_n^2}{4\theta}}\right)}{15 \times 16 \left(1 - \frac{2M}{\sqrt{\pi\theta}}e^{-\frac{M^2}{\theta}}\right)^2} \simeq 0.05$. It approaches infinity as well, although at a slower rate as shown in Fig. 4, because $\alpha \rightarrow \infty$ when $r_h \rightarrow r_0$ from $\frac{dM}{dr_h}|_{r_h=r_0} = 0$ as discussed earlier. The noncommutative entropy is larger than the noncommutative BH entropy below a critical value near M_0 determined by the curve crossing. A noncommutative black hole that evaporates down to smaller than M_0 thus can house larger information storage capacity (than the BH entropy). The crossing point shown in Fig. 4 is approximate, since the graphed volume associated entropy is estimated using (divergent) v from the second evaporation stage only.

The divergent volume of Eq. (15) is thus a general feature of noncommutative black holes irrespective of their other details. This shows noncommutative black hole can possess an infinite CR volume. It belongs to a class of black holes with finite surface [BH entropy Eq. (6)] but an infinite interior. Such a result supports the interpretation that the BH entropy might be independent of the interior of a black hole.

IV. CONCLUSION AND DISCUSSION

In conclusion we study the CR volume based on thermodynamics for Schwarzschild black hole with noncom-

mutative spacetime, which allows for a well-described final evaporation stage and results in a finite cold remnant at zero temperature. We find the CR volume is similar in expression to the result in commutative spacetime, except for a noncommutative parameter dependent modification prefactor. The improved estimate for the advanced time v leads to a divergent volume. Thus we show noncommutative Schwarzschild black hole represents a class of black hole with an infinite volume wrapped inside a finite surface.

Our work also sheds light on the information loss paradox. Assuming unitarity, regardless of the “firewall” [33], our result implies that information for a noncommutative black hole is stored on the horizon, consistent with the proposed idea of “soft hairs”, and this information can be taken away by Hawking radiations. Such a scenario has been studied by several groups [34–37], despite of the lacking for a microscopic mechanism for information transfer. If information were not taken away by radiations, the assumed unitarity becomes questionable.

Irrespective of whether radiation back reaction is included or not, we find the interior of a commutative black hole houses insufficient number of modes to account for the BH entropy as shown in Fig. 2. Introducing noncom-

mutative spacetime but neglecting the important second evaporation stage does not change such a conclusion as shown in Fig. 3. The refined treatment for the final evaporation stage by using noncommutative spacetime, on the other hand, results in a divergent volume, which shows the finite BH entropy is statistically independent of the black hole interior. The remnant in a noncommutative black hole needs an infinite time to reach, which implies that although information might be stored in the interior of a black hole, in order to preserve unitarity the complete process of evaporation needs an infinite time. This seemingly unpleasant outcome essentially heralds the breakdown of unitarity if information were housed in the black hole interior and not taken away by radiations.

V. ACKNOWLEDGE

B.C.Z is supported by NSFC (No. 11374330 and No. 91636213) and by the Fundamental Research Funds for the Central Universities, China University of Geosciences (Wuhan) (No. CUG150630). The work of L.Y. is supported by MOST 2013CB922004 of the National Key Basic Research Program of China, and by NSFC (No. 11374176 and No. 91421305).

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